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A new look at degenerate Lagrangian dynamics from the viewpoint of almost-product structures

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Abstract. Singular Lagrangian systems are studied in the framework of almost-product structures. The choice of an appropriate almost-product structure permits us to obtain the dynamics. The relationship with the Dirac bracket is also elucidated.

1. Introduction

The study of singular Lagrangian systems goes back to the seminal works of Dirac and Bergmann (see [6, 24, 25, 3]). Their algorithm was later globalized by Gotay and Nester [9, 10, 11], who introduced to the game the crucial role played by the almost-tangent structure of the phase space of velocities (see also the papers by Klein [15] and Grifone [14]). In fact, besides the ambiguity in the dynamics, the equations of motion have to be of second order.

The aim of this paper is to take a new look at degenerate Lagrangian systems from the geometrical point of view of almost-product structures. Roughly speaking, an almost-product structure on a manifold M consists of two complementary distributions on M. Therefore, if $L : TQ \longrightarrow \mathbb{R}$ is a singular Lagrangian function, it is quite natural to take an almost-product structure on TQ such that one of the two complementary distributions is just the singular distribution $\ker\omega_L$. The 'projection' of the system onto the regular distribution would give a 'regular' system with a completely determined dynamics. This approach is an alternative way of considering a quotient by the characteristic distribution $\ker\omega_L$ as studied by Cantrijn *et al* [5].

The use of almost-product structures in order to obtain the 'true' dynamics of the singular system was proposed in several recent papers (see de León and Rodrigues [17, 19], Pitanga [21, 22], Dubrovin *et al* [7] and references therein). In this paper, our approach is as follows. Consider a singular Lagrangian function $L : TQ \longrightarrow \mathbb{R}$ with presymplectic form ω_L and denote by M_1 the manifold of primary constraints. Assume for simplicity that there

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are no secondary constraints. If all the primary constraints are second class, then an almost-product structure on the phase space of momenta T^*Q may be defined and the projection of a Hamiltonian vector field corresponding to any extended Hamiltonian function gives the dynamics. Notice that the almost-product structure is closely related to the Dirac bracket. On the other hand, if all the constraints are first class, the dynamics is determined up to the choice of an almost-product structure on M_1 adapted to the presymplectic form ω_1, where ω_1 is the restriction of the canonical symplectic form ω_Q. Of course, if there are constraints of first and second class, we can combine both procedures. We remark that the role played by the second-class constraints is very similar to non-holonomic constraints in mechanics (see Pitanga [21]). In the mixed case we have to take two almost-product structures, one of them on T^*Q and the other one on M_1. If there are secondary constraints, we can perform a similar procedure with minor changes.

The paper is structured as follows. Section 2 is devoted to recalling some results on almost-product structures adapted to presymplectic structures. In fact, a Poisson bracket may be defined in a presymplectic manifold by using an adapted almost-product structure. In section 3 we recall the Dirac–Bergmann–Gotay–Nester algorithm. The case of Lagrangian systems admitting a global dynamics is considered in section 4. Suitable almost-product structures are defined in order to fix the dynamics and our procedure is compared with the 'classical' ones. The case of Lagrangian systems with secondary constraints is studied in section 5. The Lagrangian and Hamiltonian formalisms are related by using Legendre-projectable almost-product structures on the velocity space in section 6, and the second-order differential equation problem is solved in section 7. The special case of affine Lagrangians on the velocities is studied in section 8. This type of Lagrangians deserves special features since the adapted almost-product structures coincide with the connections in the sense of Ehresmann. Several examples are studied throughout the paper in order to illustrate our procedure.

2. Almost-product structures adapted to presymplectic forms

In this section we recall some definitions and results on almost-product structures (see [18] and [7]).

An almost-product structure on a manifold M is a tensor field F of type $(1, 1)$ on M such that $F^2 = \mathrm{id}$. The manifold M will be called an almost-product manifold (see [18])

If we set:

$$\mathcal{A} = \tfrac{1}{2}(\mathrm{id}+F) \qquad \mathcal{B} = \tfrac{1}{2}(\mathrm{id}-F)$$

then $\mathcal{A}$ and $\mathcal{B}$ are complementary projectors, i.e. $\mathcal{A} + \mathcal{B} = \mathrm{id}$, $\mathcal{A}^2 = \mathcal{A}$, $\mathcal{B}^2 = \mathcal{B}$, $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A} = 0$.

We denote by $\operatorname{Im}\mathcal{A}$ and $\operatorname{Im}\mathcal{B}$ the corresponding complementary distributions. Hence $TM = \operatorname{Im}\mathcal{A} \oplus \operatorname{Im}\mathcal{B}$. We denote by $\mathcal{A}^*$ and $\mathcal{B}^*$ the transpose operators and $\operatorname{Im}\mathcal{A}^*$ and $\operatorname{Im}\mathcal{B}^*$ will be their corresponding images.

Definition 2.1. Let (M, ω) be a presymplectic manifold with a presymplectic form ω. An almost-product structure F on M is said to be adapted to ω if

$$\ker\omega = \ker\mathcal{A}\,.$$

Define the linear mapping $\flat : \mathfrak{X}(M) \longrightarrow \wedge^1(M)$ by $\flat(X) = i_X\omega$. If F is adapted to ω, the restriction of $\flat$ to the distribution $\mathcal{A}$ induces an isomorphism $\flat : \operatorname{Im}\mathcal{A} \longrightarrow \operatorname{Im}\mathcal{A}^*$ of C^∞-modules.

Then, for an arbitrary 1-form α the equation

$$i_X\omega = \mathcal{A}^*\alpha \tag{1}$$

admits a unique solution $X_{\alpha,\mathcal{A}}$ such that $X_{(\alpha,\mathcal{A})} \in \operatorname{Im}\mathcal{A}$. For a function f on M we put $X_{f,\mathcal{A}} = X_{\mathrm{d}f\mathcal{A}}$. Now, we define a bracket of functions as follows:

$$\{f, g\}_{\mathcal{A}} = \omega(X_{f,\mathcal{A}}, X_{g,\mathcal{A}})$$

where $f, g \in C^\infty(M)$: $\{\ ,\ \}_{\mathcal{A}}$ satisfies all the properties of a Poisson bracket except the Jacobi identity, i.e.

(i) $\{af, g\}_{\mathcal{A}} = a\{f, g\}_{\mathcal{A}}$, for all $a \in \mathbb{R}$
(ii) $\{f + g, h\}_{\mathcal{A}} = \{f, h\}_{\mathcal{A}} + \{g, h\}_{\mathcal{A}}$
(iii) $\{f, g\}_{\mathcal{A}} = -\{g, f\}_{\mathcal{A}}$
(iv) $\{f, gh\}_{\mathcal{A}} = \{f, g\}_{\mathcal{A}}h + g\{f, h\}_{\mathcal{A}}$

for all $f, g, h \in C^\infty(M)$.

We are going to prove that the Jacobi identity is equivalent to the integrability of the almost-product structure F (see Dubrovin *et al* [7]). Let us recall that an almost-product structure F is said to be integrable if both distributions $\operatorname{Im}\mathcal{A}$ and $\operatorname{Im}\mathcal{B}$ are integrable. In our case, the distribution $\operatorname{Im}\mathcal{B} = \ker\omega$ is always integrable, but $\operatorname{Im}\mathcal{A}$ is not necessarily so. We first prove the following lemma:

Lemma 2.1.

$$i_{X_{\{f,g\}_{\mathcal{A}},\mathcal{A}}}\omega(Z) + i_{[X_{f,\mathcal{A}},X_{g,\mathcal{A}}]}\omega(Z) = \mathcal{B}^*(\mathrm{d}g)[X_{f,\mathcal{A}}, \mathcal{A}Z] - \mathcal{B}^*(\mathrm{d}f)[X_{g,\mathcal{A}}, \mathcal{A}Z]$$
$$\forall\, Z \in \mathfrak{X}(M) \quad \forall\, f, g \in C^\infty(M)\,.$$

Proof. The proof follows by straighforward computation. □

Proposition 2.1. The bracket $\{\ ,\ \}_{\mathcal{A}}$ defined by the almost-product structure F satisfies the Jacobi identity if and only if the almost-product structure F is integrable.

Proof. If $\{\ ,\ \}_{\mathcal{A}}$ satisfies the Jacobi identity then

$$X_{\{f,g\}_{\mathcal{A}}} = [X_{(g,\mathcal{A})}, X_{(f,\mathcal{A})}]$$

for any two functions f and g on M. Since the vector fields $X_{(f,\mathcal{A})}$ span $\operatorname{Im}\mathcal{A}$ thus $\operatorname{Im}\mathcal{A}$ is integrable. Therefore, the almost-product structure F is integrable (see [18]).

Conversely, if F is integrable then from lemma 2.1 we deduce that:

$$i_{X_{\{f,g\}_{\mathcal{A}}}}\omega(Z) = i_{[X_{(g,\mathcal{A})},X_{(f,\mathcal{A})}]}\omega(Z)\,.$$

Therefore, the vector fields $X_{\{f,g\}_{\mathcal{A}}}$ and $[X_{(f,\mathcal{A})}, X_{(g,\mathcal{A})}]$ differ by an element of $\ker\omega$. But, since the almost-product structure F is integrable, we deduce that $[X_{(f,\mathcal{A})}, X_{(g,\mathcal{A})}] \in \operatorname{Im}\mathcal{A}$. Thus, $X_{\{f,g\}_{\mathcal{A}},\mathcal{A}} = [X_{(g,\mathcal{A})}, X_{(f,\mathcal{A})}]\,.$ □

As a consequence, if we assume that F is integrable, we have a Poisson manifold $(M, \{\ ,\ \}_{\mathcal{A}})$ whose symplectic foliation is just $\operatorname{Im}\mathcal{A}$. Furthermore, the symplectic form on each leaf $\mathcal{L}$ is just the restriction of the presymplectic form to $\mathcal{L}$. If we denote by $\sharp_{\mathcal{A}} : T^*M \longrightarrow TM$ the linear mapping defined by $\langle\sharp_{\mathcal{A}}(\mathrm{d}f), \mathrm{d}g\rangle = \{g, f\}_{\mathcal{A}}$, then $X_{f,\mathcal{A}} = \sharp_{\mathcal{A}}(\mathrm{d}f)$. Thus, $X_{H,\mathcal{A}}f = \{f, H\}_{\mathcal{A}}$, for any function f, where $H : M \longrightarrow \mathbb{R}$ is a Hamiltonian function.

Remark 2.1. In [7] an almost-product structure was called a generalized connection. The justification of this name is the following. Suppose that $\ker\omega$ defines a regular foliating distribution, i.e. it is well-defined the quotient manifold $\bar{M} = M/\ker\omega$ and we have a fibred manifold $\pi : M \longrightarrow \bar{M}$. Hence $\mathrm{Im}\,\mathcal{A}$ defines a connection in π in the sense of Ehresmann since $\ker T\pi = \ker\omega$.

Remark 2.2. The existence of integrable almost-product structures on manifolds is a very difficult problem. A recent work on that topic is the paper by Gil *et al* [8] (see also [23, 20]). They proved that the space of all almost-product structures which are adapted to a foliation $\mathcal{F}$ on a manifold M is an analytic real manifold (of infinite dimension). The problem now is to identify which of them are integrable.

3. The constraint algorithm

Let Q be an m-dimensional manifold. Denote by $\tau_Q : TQ \longrightarrow Q$ the canonical projection. If (q^A), $1 \leqslant A \leqslant m$ are local coordinates on a neighbourhood U of Q, we denote by $(q^A, \dot{q}^A)$, $1 \leqslant A \leqslant m$, the induced coordinates on TU.

Consider a Lagrangian $L : TQ \longrightarrow \mathbb{R}$ such that the Hessian matrix

$$\left(\frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B}\right)$$

is not regular. This type of Lagrangian is called singular or degenerate. Let E_L be the energy associated with L, defined by $E_L = CL - L$, where C is the Liouville vector field on TQ. We denote by α_L the Poincaré–Cartan 1-form defined by $\alpha_L = J^*(\mathrm{d}L)$ and, by ω_L the Poincaré–Cartan 2-form given by $\omega_L = -\mathrm{d}\alpha_L$, where J is the canonical almost-tangent structure on TQ. We obtain a presymplectic system (TQ, ω_L, E_L) and ω_L is supposed to be of constant rank. In the regular case, ω_L is symplectic and then the equation of motion

$$i_X\omega_L = \mathrm{d}E_L \tag{2}$$

has a unique solution ξ_L, the Euler–Lagrange vector field; moreover, ξ_L is a second-order differential equation (SODE for brevity), that is, $J\xi_L = C$. In the degenerate case, (2) has no solution, in general, and even if it exists it will be neither unique nor a SODE.

The Legendre map $Leg : TQ \longrightarrow T^*Q$ is locally written as

$$Leg : (q^A, \dot{q}^A) \rightsquigarrow (q^A, p_A)$$

where $p_A = \partial L/\partial \dot{q}^A$ are the generalized momenta. We suppose that L is almost regular, i.e. $M_1 = Leg(TQ)$ *is a submanifold of* T^*Q *and* Leg *is a submersion onto* M_1 with connected fibres. In particular, this implies that the Hessian matrix is of constant rank. We denote by $Leg_1 : TQ \longrightarrow M_1$ the restriction of $Leg : TQ \longrightarrow T^*Q$ to its image. The *submanifold* M_1 *will be called the primary constraint submanifold.* Moreover, we have that $\ker TLeg = \ker\omega_L \cap V(TQ)$. If the Lagrangian is almost regular, the energy E_L is constant along the fibres of Leg. Therefore, E_L projects onto a function h_1 on M_1, i.e. $h_1(Leg(x)) = E_L(x)$, $\forall\, x \in TQ$.

Let λ_Q be the Liouville 1-form and $\omega_Q = -\mathrm{d}\lambda_Q$ the canonical symplectic form on T^*Q. Since ω_Q is symplectic, we have a Poisson bracket on T^*Q defined by $\{F, G\} = \omega_Q(X_F, X_G)$, $\forall\, F, G \in C^\infty(T^*Q)$. If we denote by $i : M_1 \longrightarrow T^*Q$ the natural embedding of M_1 into T^*Q, then we obtain a presymplectic system (M_1, ω_1, h_1), where $\omega_1 = i^*\omega_Q$.

There appear $m - k$ independent constraints ϕ^A which describe M_1; they are the primary constraints, following the Dirac terminology (see [6]). Notice that if M_1 is a closed submanifold, then it is generated by a set of globally defined functions (see for instance [13]). If H is an arbitrary extension of h_1 to T^*Q, all the Hamiltonian functions of the form

$$\tilde{H} = H + \lambda_A \phi^A \tag{3}$$

where λ_A are Lagrange multipliers, are weakly equal, that is, $\tilde{H}_{/M_1} = H_{/M_1} = h_1$. The Hamilton equations of the motion are written in terms of the canonical Poisson bracket of T^*Q as follows:

$$\frac{\mathrm{d}q^A}{\mathrm{d}t} = \{q^A, \tilde{H}\} \qquad \frac{\mathrm{d}p_A}{\mathrm{d}t} = \{p_A, \tilde{H}\} \qquad \phi^A = 0\,.$$

This shows that there exists an ambiguity in the description of the dynamics. Since ω_Q is symplectic, a solution of the equation $i_X\omega_Q = \mathrm{d}H$ always exists. The constraints must be preserved in the time or, equivalently, the solution X must be tangent to M_1. Then we get

$$\left(\{\phi^B, \tilde{H}\} + \lambda_A\{\phi^B, \phi^A\}\right)_{/M_1} = 0\,.$$

The vanishing of these expressions can lead two kinds of consequences: some of the arbitrary functions λ_A may be determined or new constraints may arise. These new constraints are called secondary constraints. The primary and secondary constraints define the submanifold M_2.

Now, we can proceed in a similar way with the secondary constraints, because they should also be conserved in time. This process may be continued and if the initial problem is solvable, we arrive at some final constraint submanifold M_f where 'consistent' solutions exist.

It is possible to give a classification of the constraints generated by this algorithm in order to clarify the ambiguity of the dynamics. A constraint ϕ of M_i (the i-ary constraint submanifold) is said to be first class if $\{\phi, \phi^A\}_{/M_1} = 0$ for each constraint ϕ^A of M_i, and second class otherwise. Then, the coefficients of the primary first-class constraints on M_f in (3) are completely undetermined, while the coefficients of the primary second-class constraint are completely fixed.

For a more geometric point of view, the Gotay–Nester algorithm globalizes the Dirac–Bergmann algorithm (see [9, 4]). The Gotay–Nester algorithm is applicable in more general situations than the Dirac constraint algorithm. In fact, in [9], they develop a constraint algorithm for a generic presymplectic system (S, ω, H). They consider the points of S where

$$i_X\omega = \mathrm{d}H \tag{4}$$

has a solution and suppose that this set S_2 is a submanifold of S. Nevertheless, the solutions of (4) on S_2 are not necessarily tangent to S_2. Hence, we consider the points of S_2 on which there exists a solution which is tangent to S_2. Thus, a new submanifold S_3 is obtained and the proccess may be continued. We have the following sequence of submanifolds:

$$\cdots \to S_k \to \cdots \to S_2 \to S_1 = S\,.$$

Alternatively, these submanifolds can be described as follows:

$$S_i = \{x \in S \mid v(H) = 0\,,\ \forall\, v \in T_xS_{i-1}^{\perp}\}$$

where

$$T_xS_{i-1}^{\perp} = \{v \in T_xS \mid \omega(x)(u, v) = 0\ \ ,\ \forall\, u \in T_xS_{i-1}\}\,.$$

We call S_2 the secondary constraint submanifold, S_3 the tertiary constraint submanifold, and, in general, S_i is the i-ary constraint submanifold. If the algorithm stabilizes, that is, there exists a positive integer k such that $S_k = S_{k+1}$ and $\dim S_k > 0$, then we have a final submanifold S_f where, by construction, a solution X on S_f exists, i.e. $X \in \mathfrak{X}(S_f)$ verifies that

$$(i_X\omega = \mathrm{d}H)_{/S_f} \,. \tag{5}$$

The Gotay–Nester algorithm generalizes the Dirac constraint algorithm when we consider the particular presymplectic system (M_1, ω_1, h_1). In [9, 10], Gotay and Nester have proved that the presymplectic systems (TQ, ω_L, E_L) and (M_1, ω_1, h_1) are equivalent, that is, both descriptions, Lagrangian and Hamiltonian ones, are related by the Legendre transformation.

4. Lagrangian systems with a global dynamics

First, we suppose that the presymplectic system (TQ, ω_L, E_L) admits a global dynamics, i.e. there exists at least a vector field ξ on TQ such that ξ satisfies the equation of the motion $i_\xi\omega_L = \mathrm{d}E_L$. In such a case, the submanifold $M_1 = Leg(TQ)$ of T^*Q is the final constraint submanifold or, in other words, there are no secondary constraints.

We distinguish three particular cases:

(i) all the primary constraints are second class,
(ii) all are first class, and
(iii) there exist first- and second-class constraints.

4.1. All the primary constraints are second class

We denote by Φ^A, $1 \leqslant A \leqslant s$, the constraints of M_1. The matrix with elements $C^{AB} = \{\Phi^A, \Phi^B\}$ is non-singular on M_1 and, in the sequel, we assume for simplicity that this matrix is non-singular in the entire phase space T^*Q. This matrix is also skewsymmetric and, then, the number of second-class constraints is even. We denote by (C_{AB}) its inverse matrix.

As in [2], we consider the smooth distribution D generated by the vector fields X_{Φ^A}. A direct computation shows that

$$D^\perp(x) = \{v \in T_xT^*Q \ / \ \omega_Q(x)(v, w) = 0 \ \ \forall\, w \in D(x)\} = T_xM_1 \qquad \forall\, x \in M_1 \,.$$

Let $\mathcal{Q} : D \oplus D^\perp \longrightarrow D$ be the projection onto D along $D^\perp$ and $\mathcal{P} = \mathrm{id} - \mathcal{Q}$. The projector $\mathcal{Q}$ is given by

$$\mathcal{Q} = C_{AB}X_{\Phi^A} \otimes d\Phi^B \,.$$

Take the 2-form $\Omega_D = \mathcal{P}^*\omega_Q$ (that is, $\mathcal{P}^*\omega_Q(X, Y) = \omega_Q(\mathcal{P}X, \mathcal{P}Y)$). Ω_D is a presymplectic form with constant rank $2m - s$. Moreover, the almost-product structure $(\mathcal{P}, \mathcal{Q})$ is adapted to Ω_D, that is, $\ker\mathcal{P} = \ker\Omega_D = D$. Thus, we can define a bracket $\{F, G\}_D$, called the Dirac bracket, on T^*Q as follows:

$$\begin{aligned}\{F, G\}_D &= \Omega_D(X_F, X_G) = \omega_Q(\mathcal{P}X_F, \mathcal{P}X_G)\\ &= \omega_Q(X_F - C_{AB}\{\Phi^B, F\}X_{\Phi^A}, X_G - C_{A'B'}\{\Phi^{B'}, G\}X_{\Phi^{A'}})\\ &= \{F, G\} - \{F, \Phi^A\}C_{AB}\{\Phi^B, G\}\,.\end{aligned}$$

Consider now the projected Hamiltonian function $h_1 : M_1 \longrightarrow \mathbb{R}$ defined by $h_1 \circ Leg = E_L$. We can extend h_1 to a function H on a neigborhood U of T^*Q and the Dirac theory argues that the Hamiltonians on U should be of the form

$$\tilde{H} = H + \lambda_A \Phi^A .$$

We consider the Hamiltonian vector field $X_{\tilde{H}}$. The consistency of the theory demands that the constraints Φ_A be preserved by $X_{\tilde{H}}$; geometrically this means that the vector field $X_{\tilde{H}}$ must be tangent to M_1.

Consider the vector field

$$\mathcal{P}X_H = X_H - C_{AB}\{\Phi^B, H\}X_{\Phi^A} .$$

By definition of the almost-product structure $(\mathcal{P}, \mathcal{Q})$, $\mathcal{P}X_H$ is tangent to M_1 and its restriction to M_1, $\mathcal{P}X_{H/M_1}$, is the unique solution of the equations of motion, that is,

$$i_{\mathcal{P}X_{H/M_1}} \omega_1 = \mathrm{d}h_1$$

because ω_1 is symplectic. Moreover, if the distribution D is integrable, then the Dirac bracket $\{\ ,\ \}_D$ is in fact a Poisson bracket. In that case, when we consider on M_1 the Poisson bracket $\{\ \}_1$ defined by the symplectic structure ω_1 and on T^*Q the Dirac bracket $\{\ ,\ \}_D$, we get that the canonical embedding $i : M_1 \longrightarrow T^*Q$ is a Poisson morphism, that is,

$$i^*\{F, G\}_D = \{i^*F, i^*G\}_1 \qquad \forall\, F, G \in C^\infty(T^*Q) .$$

The above results are summarized in table 1.

Table 1. Second-class primary constraints

T^*Q	T^*Q	M_1
ω_Q	Ω_D	ω_1
$\{\ ,\ \}$	$\{\ ,\ \}_D$	$\{\ ,\ \}_1$
	$(\mathcal{P}, \mathcal{Q})$	

Example 4.1. Let $L : T\mathbb{R}^4 \longrightarrow \mathbb{R}$ be the Lagrangian defined by (see [1])

$$L(q^A, \dot{q}^A) = (q^2 + q^3)\dot{q}^1 + q^4\dot{q}^3 + \tfrac{1}{2}((q^4)^2 - 2q^2q^3 - (q^3)^2) .$$

Since

$$qp_1 = \frac{\partial L}{\partial \dot{q}^1} = q^2 + q^3 \qquad p_2 = \frac{\partial L}{\partial \dot{q}^2} = 0 \qquad p_3 = \frac{\partial L}{\partial \dot{q}^3} = q^4 \qquad p_4 = \frac{\partial L}{\partial \dot{q}^4} = 0$$

we obtain the following primary constraints:

$$\Phi_1 = p_1 - q^2 - q^3 \qquad \Phi_2 = p_2 \qquad \Phi_3 = p_3 - q^4 \qquad \Phi_4 = p_4 .$$

All of them are second-class constraints. Let C be the matrix

$$(C^{AB}) = (\{\Phi^A, \Phi^B\}) = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence, we obtain an almost-product structure $(\mathcal{P}, \mathcal{Q})$ defined by

$$\mathcal{Q} = C_{AB} X_{\Phi^A} \otimes d\Phi^B$$

or, in canonical coordinates on T^*Q

$$\begin{aligned} \mathcal{Q} = \frac{\partial}{\partial q^2} \otimes \mathrm{d}q^2 + \frac{\partial}{\partial q^2} \otimes \mathrm{d}q^3 + \frac{\partial}{\partial q^4} \otimes \mathrm{d}q^4 - \frac{\partial}{\partial q^2} \otimes \mathrm{d}p_1 \\ + \left(\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^4} + \frac{\partial}{\partial p_2} + \frac{\partial}{\partial p_3} \right) \otimes \mathrm{d}p_1 - \frac{\partial}{\partial q^4} \otimes \mathrm{d}p_3 \\ + \left(\frac{\partial}{\partial q^3} - \frac{\partial}{\partial q^2} + \frac{\partial}{\partial p_4} \right) \otimes \mathrm{d}p_4 . \end{aligned}$$

The presymplectic 2-form Ω_D is

$$\Omega_D = \mathrm{d}q^1 \wedge \mathrm{d}p_1 - \mathrm{d}q^3 \wedge \mathrm{d}p_2 + \mathrm{d}q^3 \wedge \mathrm{d}p_3 + \mathrm{d}p_1 \wedge \mathrm{d}p_2 - \mathrm{d}p_2 \wedge \mathrm{d}p_4 + \mathrm{d}p_3 \wedge \mathrm{d}p_4 .$$

The 1-form ω_1 is given in local coordinates (q^A) on M_1 by

$$\omega_1 = \mathrm{d}q^1 \wedge \mathrm{d}q^2 + \mathrm{d}q^1 \wedge \mathrm{d}q^3 + \mathrm{d}q^3 \wedge \mathrm{d}q^4$$

which is obviously a symplectic form. The unique solution ξ_{M_1} of the equation $i_X\omega_1 = \mathrm{d}h_1$ is precisely

$$\xi_{M_1} = q^3 \frac{\partial}{\partial q^1} - q^4 \frac{\partial}{\partial q^2} + q^4 \frac{\partial}{\partial q^3} - q^2 \frac{\partial}{\partial q^4} .$$

Therefore, if H is an arbitrary extension of h_1 to T^*Q then we set $\mathcal{P}(X_H)_{/M_1} = \xi_{M_1}$. The Lagrangian L is affine on the velocities. The general case will be studied in section 8.

4.2. All the primary constraints are of first class

We denote by ϕ^i, $1 \leqslant i \leqslant p$, the first-class constraints. Since $\{\phi^i, \phi^j\}_{/M_1} = 0$, then X_{ϕ_i}, $1 \leqslant i \leqslant p$, the Hamiltonian vector field of ϕ^i, is tangent to M_1. Notice that the submanifold M_1 is coisotropic into T^*Q.

Since $\ker \omega_1$ is generated by the restrictions of the Hamiltonian vector fields X_{ϕ^i} of the first-class constraints, in order to fix the gauge, we take an almost structure $(\mathcal{A}_1, \mathcal{B}_1)$ on M_1 adapted to $\ker \omega_1$. Moreover, if the almost-product structure is integrable, we can define a Poisson bracket on M_1 as follows:

$$\{f, g\}_{\mathcal{A}_1} = \omega_1(X_{f,\mathcal{A}_1}, X_{g,\mathcal{A}_1}) \qquad \forall\ f, g \in C^\infty(M_1)$$

where $X_{f,\mathcal{A}_1}$ and $X_{g,\mathcal{A}_1}$ are the unique vector fields on M_1 which belong to $\mathrm{Im}\,\mathcal{A}_1$ and verify that $i_{X_{f,\mathcal{A}_1}}\omega_1 = \mathcal{A}_1^*\mathrm{d}f$ and $i_{X_{g,\mathcal{A}_1}}\omega_1 = \mathcal{A}_1^*\mathrm{d}g$, respectively. Therefore, if ξ is a solution of the equation of motion, that is, $i_\xi \omega_1 = dh_1$, we can select a unique solution $\mathcal{A}_1\xi$ such that $\mathcal{A}_1\xi \in \operatorname{Im} \mathcal{A}_1$. Thus we have fixed the gauge.

The results of this section are summarized in table 2.

Table 2. First-class primary constraints.

T^*Q	M_1
ω_Q	ω_1
$\{\, ,\}$	$\{\, ,\}_{\mathcal{A}_1}$
	$(\mathcal{A}_1, \mathcal{B}_1)$

Now, consider an arbitrary extension H to T^*Q of the Hamiltonian $h_1 : M_1 \longrightarrow \mathbb{R}$. Since we have a global dynamics, the Hamiltonian vector field X_H is tangent to M_1, i.e. $X_{H/M_1} \in \mathfrak{X}(M_1)$ and

$$i_{X_{H/M_1}} \omega_1 = dh_1 .$$

We fix the gauge by taking $\mathcal{A}_1(X_{H/M_1})$.

The classical procedure is the following (see [27]). Choose functions $\{f^j,\ 1 \leqslant j \leqslant p\}$, on T^*Q such that the matrix $(\{\phi^i, f^j\}) = (c^{ij})$ is regular. The determinant of this matrix is called the Fadeev–Popov determinant. If we impose the tangency of the Hamiltonian vector fields of the Hamiltonian functions $\tilde{H} = H + \lambda_i\phi^i$ to the submanifold defined by the new constraints $\{f^j\}$, we get that

$$\lambda^i{}_{/M_1} = \left(c_{ij}\{H, \phi^j\}\right)_{/M_1} .$$

Thus we have fixed the gauge. It is easy to prove that fixing the gauge is equivalent to take an almost-product structure $(\mathcal{P}, \mathcal{Q})$ on T^*Q where

$$\mathcal{Q} = c_{ij} X_{\phi^i} \otimes df^j .$$

The almost-product structure $(\mathcal{P}, \mathcal{Q})$ restricts to M_1 and this restriction $(\mathcal{P}_{/M_1}, \mathcal{Q}_{/M_1})$ is adapted to ω_1. Thus

$$\mathcal{P}(X_H)_{/M_1} = X_{h_1, \mathcal{P}_{/M_1}} .$$

Example 4.2. Consider the Lagrangian function $L : T\mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by

$$L = \tfrac{1}{2}(\dot{q}_1 + \dot{q}_2)^2 .$$

(see Krupková [16]). Here (q^1, q^2, q^3) are the standard coordinates on $\mathbb{R}^3$ and $(q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \dot{q}^3)$ the induced ones on $T\mathbb{R}^3$.

The energy and the Poincaré–Cartan forms are

$$E_L = \tfrac{1}{2}(\dot q_1 + \dot q_2)^2 = L$$
$$\alpha_L = (\dot q_1 + \dot q_2)\mathrm{d}q_1 + (\dot q_1 + \dot q_2)\mathrm{d}q_2$$
$$\omega_L = \mathrm{d}q_1 \wedge \mathrm{d}\dot q_1 + \mathrm{d}q_1 \wedge \mathrm{d}\dot q_2 + \mathrm{d}q_2 \wedge \mathrm{d}\dot q_1 + \mathrm{d}q_2 \wedge \mathrm{d}\dot q_2 \,.$$

There are no secondary constraints, i.e. we have a global dynamics. Since

$$p_1 = \frac{\partial L}{\partial \dot q^1} = \dot q^1 + \dot q^2 \qquad p_2 = \frac{\partial L}{\partial \dot q^2} = \dot q^1 + \dot q^2 \qquad p_3 = \frac{\partial L}{\partial \dot q^1} = 0$$

we deduce that the submanifold M_1 of $T^*\mathbb{R}^3$ is defined by the following primary constraints:

$$\phi_1 = p_1 - p_2 = 0 \qquad \phi_2 = p_3 = 0\,.$$

Since $\{\phi_1, \phi_2\} = 0$, then both constraints are first class. If we take coordinates (q^1, q^2, q^3, p_1) on M_1, we obtain that

$$\omega_1 = i^*\omega_Q = \mathrm{d}q^1 \wedge \mathrm{d}p_1 + \mathrm{d}q^2 \wedge \mathrm{d}p_1$$

where

$$i(q^1, q^2, q^3, p_1) = (q^1, q^2, q^3, p_1, p_1, 0)\,.$$

Thus, $\ker\omega_1$ is generated by

$$\left\{ \frac{\partial}{\partial q^3}\,,\ \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2} \right\}$$

The almost-product structure $(\mathcal{A}_1, \mathcal{B}_1)$ on M_1 defined by

$$\mathcal{A}_1\left(\frac{\partial}{\partial q^1}\right) = \frac{\partial}{\partial q^1} \qquad \mathcal{A}_1\left(\frac{\partial}{\partial q^2}\right) = \frac{\partial}{\partial q^1} \qquad \mathcal{A}_1\left(\frac{\partial}{\partial q^3}\right) = 0 \qquad \mathcal{A}_1\left(\frac{\partial}{\partial p^1}\right) = \frac{\partial}{\partial p^1}$$

where $\mathcal{B}_1 = \mathrm{id} - \mathcal{A}_1$; $(\mathcal{A}_1, \mathcal{B}_1)$ is integrable and adapted to ω_1. Then it defines a Poisson bracket $\{\ ,\ \}_{M_1}$ on M_1. Since

$$X_{q^1,\mathcal{A}^1} = -\frac{\partial}{\partial p^1} \qquad X_{q^2,\mathcal{A}^1} = -\frac{\partial}{\partial p^1}$$
$$X_{q^3,\mathcal{A}^1} = 0 \qquad X_{p^1,\mathcal{A}^1} = \frac{1}{2}\left(\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}\right)$$

we get

$$\{q^1, q^2\}_{\mathcal{A}_1} = 0 \qquad \{q^1, q^3\}_{\mathcal{A}_1} = 0 \qquad \{q^2, q^3\}_{\mathcal{A}_1} = 0$$
$$\{q^1, p_1\}_{\mathcal{A}_1} = -1 \qquad \{q^2, p_1\}_{\mathcal{A}_1} = -1 \qquad \{q^3, p_1\}_{\mathcal{A}_1} = 0\,.$$

If ξ is a vector field on M_1 which is a solution of the equations of motion, i.e. $i_\xi\omega_1 = \mathrm{d}h_1$, then we fix a unique solution by taking $\mathcal{A}_1(\xi) = X_{h_1,\mathcal{A}_1}$. In that case, we have

$$X_{h_1,\mathcal{A}_1} = p_1\frac{\partial}{\partial q^1} + p_1\frac{\partial}{\partial q^2}\,.$$

4.3. There exist first- and second-class constraints

We denote by Φ^A, $1 \leqslant A \leqslant s$ the second-class constraints and by ϕ^i, $1 \leqslant i \leqslant p$, the first-class constraints.

As in the first case, we can construct an almost-product structure $(\mathcal{P}, \mathcal{Q})$ on T^*Q with $\mathcal{Q}$ given by

$$\mathcal{Q} = C_{AB} X_{\Phi^A} \otimes d\Phi^B .$$

Here, (C_{AB}) is the inverse matrix of $(\{\Phi^A, \Phi^B\})$. We also have the presymplectic form $\Omega_D = \mathcal{P}^* \omega_Q$ with constant rank $2m - s$ and the Dirac bracket

$$\{F, G\}_D = \{F, G\} - \{F, \Phi^A\} C_{AB} \{\Phi^B, G\} .$$

If we consider an arbitrary extension H of the Hamiltonian h_1, since we have a global dynamics, then the vector field $\mathcal{P}(X_H)$ is tangent to M_1. Now, we fix the gauge taking the vector field $\mathcal{A}_1(\mathcal{P}(X_H)_{/M_1})$, where $(\mathcal{A}_1, \mathcal{B}_1)$ is some almost-product structure adapted to ω_1.

The the results of this section are summarized in table 3.

Table 3. First- and second-class primary constraints.

T^*Q	T^*Q	M_1
ω_Q	Ω_D	ω_1
$\{\,,\,\}$	$\{\,,\,\}_D$	$\{\,,\,\}_{\mathcal{A}_1}$
	$(\mathcal{P}, \mathcal{Q})$	$(\mathcal{A}_1, \mathcal{B}_1)$

5. Lagrangian systems with secondary constraints

We denote by $\{\Phi^A, \phi^i\ ;\ 1 \leqslant A \leqslant s\ ,\ 1 \leqslant i \leqslant p\}$ the set of primary second- and first-class constraints.

We apply the Gotay–Nester algorithm to the presymplectic system (M_1, ω_1, h_1) and we obtain a sequence of submanifolds

$$M_f \longrightarrow \cdots \longrightarrow M_k \longrightarrow M_{k-1} \longrightarrow \cdots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow T^*Q .$$

Here, we suppose that the algorithm stabilizes, that is, there exists a positive integer k such that $M_{k+1} = M_k$ and $\dim M_k > 0$. We denote by M_f the final constraint submanifold. The constraint submanifold will be determined by all the primary and secondary constraints (for simplicity, we call secondary constrain each constraint which is not primary). Now, we can classify all these constraints of M_f in two classes: first-class constraints and second-class. We denote by $\{\bar{\Phi}^B\ ,\ 1 \leqslant B \leqslant \bar{s}\}$ the secondary second-class constraints and by $\{\bar{\phi}^j\ ,\ 1 \leqslant j \leqslant \bar{p}\}$ the secondary first-class constraints. The primary second-class constraints of M_1 are also second-class on M_f but the primary first-class constraints may be first or second class on M_f. Then, we can suppose that $\{\phi^{i'}, 1 \leqslant i' \leqslant p'\}$ are primary first-class constraints which are also first class on M_f and $\{\phi^{i''}, 1 \leqslant i'' \leqslant p''\}$ are primary first-class constraints which are second class on M_f, where $p' + p'' = p$. We then have the following classification of constraints on M_f:

$\{\phi^{i'}, \bar{\phi}^j, 1 \leqslant i' \leqslant p', 1 \leqslant j \leqslant \bar{p}\}$ first-class constraints on M_f

$\{\phi^{i''}, \Phi^A, \bar{\Phi}^B, 1 \leqslant i'' \leqslant p'', 1 \leqslant A \leqslant s, 1 \leqslant B \leqslant \bar{s}\}$

second-class constraints on M_f.

We denote by $\{\chi^\alpha\}$ all the second-class constraints on M_f which are used to define an almost-product structure $(\mathcal{P}, \mathcal{Q})$ on T^*Q, where the projector $\mathcal{Q}$ is defined as follows:

$$\mathcal{Q} = C_{\alpha\beta} X_{\chi^\alpha} \otimes \mathrm{d}\chi^\beta$$

$(C_{\alpha\beta})$ being the inverse of the matrix $(\{\chi^\alpha, \chi^\beta\})$. As in subsection 4.1, we define the Dirac bracket

$$\{F, G\}_D = \{F, G\} - \{F, \chi^\alpha\} C_{\alpha\beta} \{\chi^\beta, G\}$$

for any functions $F, G \in C^\infty(T^*Q)$.

We now consider a local extension H of h_1 to T^*Q. Take the vector field X_H and its projection $\mathcal{P}(X_H)$. Then, $\mathcal{P}(X_H)$ is tangent to M_f, and, moreover, it is a solution of the equation of motion, that is

$$\left(i_{\mathcal{P}(X_H)_{/M_f}} \omega_1 = \mathrm{d}h_1\right)_{/M_f} .$$

In order to fix the gauge, consider an almost-product structure $((\mathcal{A}_1)_f, (\mathcal{B}_1)_f)$ on M_f such that it is adapted to the distribution $\ker\omega_1 \cap TM_f$, i.e. $\ker(\mathcal{B}_1)_f = \ker\omega_1 \cap TM_f$. Now, it is sufficient to take $(\mathcal{A}_1)_f(\mathcal{P}(X_H)_{/M_f})$ and the gauge will remain fixed. If we consider the 'extended' Hamiltonian (an extension of the Hamiltonian h_1 where we have account of all the constraints of M_f, see [12]), then it is convenient to use, in order to fix the gauge, an almost-product structure $(\mathcal{A}_f, \mathcal{B}_f)$ on M_f adapted to $\ker\omega_f$, where $\omega_f = i_f^*\omega_Q$. Here, we denote by $j_f : M_f \longrightarrow T^*Q$ the canonical embedding. We fix the gauge by taking $\mathcal{A}_f(\mathcal{P}(X_H)_{/M_f})$.

The results of this section are summarized in table 4.

Table 4. The general case.

T^*Q	T^*Q	M_f
ω_Q	Ω_D	
$\{,\}$	$\{,\}_D$	$((\mathcal{A}_1)_f, (\mathcal{B}_1)_f)$
	$(\mathcal{P}, \mathcal{Q})$	$\ker(\mathcal{B}_1)_f = \ker\omega_1 \cap TM_f$

Example 5.1. Let $L : T\mathbb{R}^2 \longrightarrow \mathbb{R}$ be the degenerate Lagrangian given by (see [4]):

$$L(q^A, \dot{q}^A) = \tfrac{1}{2}(\dot{q}^1)^2 + \tfrac{1}{2}(q^1)^2 q^2 .$$

The Legendre map is

$$Leg(q^1, q^2, \dot{q}^1, \dot{q}^2) = (q^1, q^2, \dot{q}^1, 0)$$

and we have a primary constraint $\phi^1 = p_2$. Consistency of this constraints leads to a secondary constraint: $\Phi^2 = q^1$ and, consistency of Φ^1 gives the tertiary constraint $\Phi^3 = p_1$. Since Φ^2 and Φ^3 are second-class constraints, then the projector $\mathcal{Q}$ is given by

$$\mathcal{Q} = -X_{q^1} \otimes \mathrm{d}p_1 + X_{p_1} \otimes \mathrm{d}q^1 = \frac{\partial}{\partial q^1} \otimes \mathrm{d}q^1 + \frac{\partial}{\partial p_1} \otimes \mathrm{d}p_1$$

and the Dirac bracket $\{\ ,\ \}_D$ is

$$\begin{array}{lll} \{q^1, q^2\}_D = 0 & \{q^1, p_1\}_D = 0 & \{q^1, p_2\}_D = 0 \\ \{q^2, p_1\}_D = 0 & \{q^2, p_2\}_D = 1 & \{p_1, p_2\}_D = 0 . \end{array}$$

The Hamiltonian $h_1 : M_1 \longrightarrow \mathbb{R}$ is

$$h_1 = \tfrac{1}{2}(p_1)^2 - \tfrac{1}{2}(q^1)^2 q^2 .$$

An arbitrary extension to T^*Q is given by: $H = \frac{1}{2}(p_1)^2 - \frac{1}{2}(q^1)^2 q^2 + \lambda p_2$, whose Hamiltonian vector field is

$$X_H = \left(p_1 + p_2\frac{\partial\lambda}{\partial p_1}\right)\frac{\partial}{\partial q^1} + \left(\lambda + p_2\frac{\partial\lambda}{\partial p_2}\right)\frac{\partial}{\partial q^2} + \left(q^1 q^2 - p_2\frac{\partial\lambda}{\partial q^1}\right)\frac{\partial}{\partial p_1} + \left(\frac{1}{2}(q^1)^2 - p_2\frac{\partial\lambda}{\partial q^2}\right)\frac{\partial}{\partial p_2}$$

and we then have

$$\mathcal{P}(X_H) = (\mathrm{id} - \mathcal{Q})(X_H) = \left(\lambda + p_2\frac{\partial\lambda}{\partial p_2}\right)\frac{\partial}{\partial q^2} + \left(\tfrac{1}{2}(q^1)^2 - p_2\frac{\partial\lambda}{\partial q^2}\right)\frac{\partial}{\partial p_2} .$$

The restriction of $\mathcal{P}(X_H)$ to $M_3 = \{(q^1, q^2, p_1, p_2) \in T^*\mathbb{R}^2 \ / \ q^1 = 0, p_1 = 0, p_2 = 0\}$ is precisely

$$\mathcal{P}(X_H)_{/M_3} = \lambda\frac{\partial}{\partial q^2}$$

and, therefore, the dynamics is fully undetermined.

6. Legendre projectable almost-product structures

Now, we want to relate the Lagrangian and Hamiltonian formulations when we have an almost-product structure which is Leg-projectable on TQ. For simplicity, we only consider Lagrangian systems which admit a global dynamics. A generalization for general degenerate Lagrangian systems is straightforward.

The following proposition gives a necessary and sufficient condition for an almost-product structure on TQ to be projectable onto M_1.

Proposition 6.1. Let $\tilde{F}$ be an almost-product structure which is adapted to the presymplectic 2-form ω_L. Then, $\tilde{F}$ is Leg_1-projectable onto M_1 if and only if

$$\tilde{\mathcal{B}}[Z, \tilde{\mathcal{A}}X] \in V(TQ) \qquad \forall\ Z \in \ker TLeg \quad \forall\ X \in \mathfrak{X}(TQ)$$

where $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are the projectors associated with $\tilde{F}$.

Proof. The almost-product structure $\tilde{F}$ is projectable if and only if (see [5])

(i) $\tilde{\mathcal{A}}(\ker TLeg) \subset \ker TLeg$
(ii) $\mathrm{Im}(L_Z\tilde{\mathcal{A}}) \subset \ker TLeg \forall\ Z \in \ker TLeg$.

Since $\ker TLeg = V(TQ) \cap \ker \omega_L$, then $\tilde{\mathcal{A}}(\ker TLeg) = 0$. Now, for all vector fields Y on TQ and $Z \in \ker TLeg$, we have that

$$\begin{aligned} L_Z\tilde{\mathcal{A}}(Y) &= [Z, \tilde{\mathcal{A}}Y] - \tilde{\mathcal{A}}[Z, Y] = [Z, \tilde{\mathcal{A}}Y] - \tilde{\mathcal{A}}[Z, \tilde{\mathcal{A}}Y] - \tilde{\mathcal{A}}[Z, \tilde{\mathcal{B}}Y] \\ &= [Z, \tilde{\mathcal{A}}Y] - \tilde{\mathcal{A}}[Z, \tilde{\mathcal{A}}Y] = \tilde{\mathcal{B}}[Z, \tilde{\mathcal{A}}Y] \end{aligned}$$

because $\ker \omega_L$ is an integrable distribution. □

Corollary 6.1. If the almost-product structure $\tilde{F}$ adapted to ω_L commutes with the canonical almost-tangent structure J, that is, $J\tilde{F} = \tilde{F}J$, then $\tilde{F}$ is projectable onto an almost-product structure on M_1 if and only if

$$J[Z, \tilde{\mathcal{A}}Y] \in \mathrm{Im}\,\tilde{\mathcal{A}} \qquad \forall\ Y \in \mathfrak{X}(TQ) \quad \forall\ Z \in \ker TLeg\,.$$

Let F be an almost-product structure on the configuration space Q and let F^{c} be the complete lift of F to TQ. Let us recall that F^{c} is defined by:

$$F^{\mathrm{c}}(X^{\mathrm{c}}) = (F(X))^{\mathrm{c}} \qquad F^{\mathrm{c}}(X^{\mathrm{v}}) = (F(X))^{\mathrm{v}} \qquad \forall\ X \in \mathfrak{X}(Q)$$

where X^{c} and X^{v} denote the complete and vertical lift of the vector field X, respectively. F^{c} is an almost-product structure on TQ and F^{c} is integrable if and only if F is integrable. If $\mathcal{A}$ and $\mathcal{B}$ are the corresponding projectors of F then $\mathcal{A}^{\mathrm{c}}$ and $\mathcal{B}^{\mathrm{c}}$ are the corresponding ones of F^{c}. We deduce that $\mathrm{Im}\,\mathcal{A}^{\mathrm{c}}$ is, in fact, the complete lift of the distribution $\mathrm{Im}\,\mathcal{A}$. In a similar way, we have that $\mathrm{Im}\,\mathcal{B}^{\mathrm{c}} = (\mathrm{Im}\,\mathcal{B})^{\mathrm{c}}$. These kind of distributions are called tangent in [5].

Corollary 6.2. If the almost-product structure F^{c} is adapted to ω_L then it is projectable onto M_1.

Proof. Since $JF^{\mathrm{c}} = F^{\mathrm{c}}J$, from corollary 6.1, we only need to prove that

$$J[Z, \mathcal{A}^{\mathrm{c}}Y] \in \mathrm{Im}\,\mathcal{A}^{\mathrm{c}} \qquad \forall\ Y \in \mathfrak{X}(TQ) \quad \forall\ Z \in \ker TLeg\,.$$

If $\{X_1, X_2, \cdots, X_r\}$ is a local basis of $\mathrm{Im}\,\mathcal{A}$, then $\{X_1^{\mathrm{c}}, X_2^{\mathrm{c}}, \cdots, X_r^{\mathrm{c}}, X_1^{\mathrm{v}}, X_2^{\mathrm{v}}, \cdots, X_r^{\mathrm{v}}\}$ is a local basis of $\mathrm{Im}\,\mathcal{A}^{\mathrm{c}}$. Thus, since $[Z, X_i^{\mathrm{v}}]$ and $[Z, X_i^{\mathrm{c}}]$ are vertical vector fields, for all $1 \leqslant i \leqslant r$, we get that F^{c} is projectable. □

Proposition 6.2. Let $\tilde{F}$ be an integrable almost-product structure adapted to ω_L and projectable onto M_1. Then, its projection F_1 is also integrable and adapted to ω_1.

Proof. The integrability of $\tilde{F}$ is trivial because the Nijenhuis tensor $N_{\tilde{F}}$ of $\tilde{F}$, projects onto the Nijenhuis tensor N_{F_1} of the projection F_1.

Since $Leg_1^*\omega_1 = \omega_L$ and $\ker\tilde{A} = \ker\omega_L$, then for all $\bar{Z} \in \mathfrak{X}(M_1)$ such that $i_{\bar{Z}}\omega_1 = 0$ and for all vector field $Z \in \mathfrak{X}(TQ)$ which is Leg_1-projectable onto $\bar{Z}$, i.e. $TLeg_1(Z) = \bar{Z}$, we obtain that

$$0 = Leg_1^*\left(i_{\bar{Z}}\omega_1\right) = i_Z Leg_1^*\omega_1 = i_Z\omega_L \,.$$

Thus, $Z \in \ker\omega_L = \ker\tilde{A}$. Therefore, its projection $\bar{Z} \in \ker\mathcal{A}_1$. Hence, we have proved that $\ker\omega_1 \subset \ker\mathcal{A}_1$. That $\ker\mathcal{A}_1 \subset \ker\omega_1$ is proved by a similar device. □

Next, we suppose that $\tilde{F}$ is an integrable almost-product structure on TQ which is adapted to ω_L and projectable onto M_1. We denote by $\{\ ,\ \}_{\tilde{A}}$ the Poisson bracket defined on TQ. If F_1 is the projected integrable almost-product structure on M_1, we know that F_1 is adapted to ω_1. Denote by $\{\ ,\ \}_{\mathcal{A}_1}$ the corresponding Poisson bracket on M_1. We are going to relate both Poisson brackets.

Lemma 6.1. For a function $\bar{f}$ on M_1, we have that $X_{\bar{f}\circ Leg_1,\tilde{A}}$ is projectable onto $X_{\bar{f},\mathcal{A}_1}$.

Proof. First, we prove that $X_{\bar{f}\circ Leg_1,\tilde{A}}$ is Leg_1-projectable, that is,

$$[X_{\bar{f}\circ Leg_1,\tilde{A}}, \tilde{Z}] \in \ker TLeg \qquad \forall\, \tilde{Z} \in \ker TLeg \,.$$

In fact,

$$\begin{aligned}
i_{[X_{\bar{f}\circ Leg_1,\tilde{A}},\tilde{Z}]}\omega_L &= L_{X_{\bar{f}\circ Leg_1,\tilde{A}}} i_{\tilde{Z}}\omega_L - i_{\tilde{Z}} L_{X_{\bar{f}\circ Leg_1,\tilde{A}}}\omega_L \\
&= -i_{\tilde{Z}}\mathrm{d}\left(i_{X_{\bar{f}\circ Leg_1,\tilde{A}}}\omega_L\right) \\
&= -i_{\tilde{Z}}\mathrm{d}\left(\tilde{A}^* d(\bar{f}\circ Leg_1)\right) \\
&= -i_{\tilde{Z}} Leg_1^*\mathrm{d}\left(\tilde{A}_1^* d(\bar{f})\right) \\
&= 0
\end{aligned}$$

from which we get $[X_{\bar{f}\circ Leg_1,\tilde{A}}, \tilde{Z}] \in \ker\omega_L$. From proposition 6.1, we deduce that

$$[X_{\bar{f}\circ Leg_1,\tilde{A}}, \tilde{Z}] \in V(TQ)$$

and, $X_{\bar{f}\circ Leg_1,\tilde{A}}$ is thus projectable. Moreover, since

$$i_{X_{\bar{f}\circ Leg_1,\tilde{A}}}\omega_L = \tilde{A}^* d(\bar{f}\circ Leg_1)$$

its projection $TLeg(X_{\bar{f}\circ Leg_1,\tilde{A}})$ verifies that

$$i_{TLeg(X_{\bar{f}\circ Leg_1,\tilde{A}})}\omega_{M_1} = \mathcal{A}_1^* d(\bar{f}) \,.$$

Therefore, we obtain $TLeg(X_{\bar{f}\circ Leg_1,\tilde{A}}) = TLeg(X_{\bar{f},\mathcal{A}_1})$. □

Proposition 6.3. The map $Leg_1 : TQ \longrightarrow M_1$ is a Poisson morphism, that is,

$$\{\bar{f}_1, \bar{f}_2\}_{\mathcal{A}_1} \circ Leg_1 = \{\bar{f}_1 \circ Leg_1, \bar{f}_2 \circ Leg_1\}_{\tilde{A}} \qquad \forall\, \bar{f}_1, \bar{f}_2 \in C^\infty(M_1) \,.$$

Proof. From lemma 6.1, we have that

$$\begin{aligned}\{\bar f_1, \bar f_2\}_{A_1} \circ Leg_1 &= Leg_1^*\left(\omega_{M_1}(X_{\bar f_1, A_1}, X_{\bar f_2, A_1})\right)\\ &= \omega_L(X_{\bar f_1 \circ Leg_1, \bar A}, X_{\bar f_2 \circ Leg_1, \bar A})\\ &= \{\bar f_1 \circ Leg_1, \bar f_2 \circ Leg_1\}_{\bar A}\end{aligned}$$

for any functions f_1 and f_2 on M_1. □

Example 6.1. Consider the Lagrangian function defined in example 4.2. A direct computation shows that $\ker \omega_L$ is generated by

$$\left\{\frac{\partial}{\partial q_1} - \frac{\partial}{\partial q_2}, \frac{\partial}{\partial q_3}, \frac{\partial}{\partial \dot q_1} - \frac{\partial}{\partial \dot q_2}, \frac{\partial}{\partial \dot q_3}\right\}.$$

Let F be an almost-product structure on Q defined as follows:

$$F\left(\frac{\partial}{\partial q^1}\right) = \frac{\partial}{\partial q^1}, F\left(\frac{\partial}{\partial q^2}\right) = 2\frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}, F\left(\frac{\partial}{\partial q^3}\right) = -\frac{\partial}{\partial q^3}.$$

The matrix representations of the corresponding projectors, $\mathcal{A}$ and $\mathcal{B}$, are respectively

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 0 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The complete lifts of the $(1,1)$-tensor fields $\mathcal{A}^c$ and $\mathcal{B}^c$ are:

$$\mathcal{A}^c = \begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{A}\end{pmatrix} \quad \text{and} \quad \mathcal{B}^c = \begin{pmatrix} \mathcal{B} & 0\\ 0 & \mathcal{B}\end{pmatrix}.$$

The almost-product structure F^c is integrable and adapted to the presymplectic form ω_L. Since

$$\begin{aligned}
&X_{q^1, \mathcal{A}^c} = -\frac{1}{2}\left(\frac{\partial}{\partial \dot q^1} + \frac{\partial}{\partial \dot q^2}\right) && X_{q^2, \mathcal{A}^c} = -\frac{1}{2}\left(\frac{\partial}{\partial \dot q^1} + \frac{\partial}{\partial \dot q^2}\right)\\
&X_{q^3, \mathcal{A}^c} = 0 && X_{\dot q^1, \mathcal{A}^c} = \frac{1}{2}\left(\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}\right)\\
&X_{\dot q^2, \mathcal{A}^c} = \frac{1}{2}\left(\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}\right) && X_{\dot q^3, \mathcal{A}^c} = 0
\end{aligned}$$

the Poisson bracket on TQ is given by

$$\begin{array}{lll}
\{q^1, q^2\}_{\mathcal{A}^c} = 0 & \{q^1, q^3\}_{\mathcal{A}^c} = 0 & \{q^2, q^3\}_{\mathcal{A}^c} = 0\\
\{\dot q^1, \dot q^2\}_{\mathcal{A}^c} = 0 & \{\dot q^1, \dot q^3\}_{\mathcal{A}^c} = 0 & \{\dot q^2, \dot q^3\}_{\mathcal{A}^c} = 0\\
\{q^1, \dot q^1\}_{\mathcal{A}^c} = -1 & \{q^1, \dot q^2\}_{\mathcal{A}^c} = -1 & \{q^1, \dot q^3\}_{\mathcal{A}^c} = 0\\
\{q^2, \dot q^1\}_{\mathcal{A}^c} = -1 & \{q^2, \dot q^2\}_{\mathcal{A}^c} = -1 & \{q^2, \dot q^3\}_{\mathcal{A}^c} = 0\\
\{q^3, \dot q^1\}_{\mathcal{A}^c} = 0 & \{q^3, \dot q^2\}_{\mathcal{A}^c} = 0 & \{q^3, \dot q^3\}_{\mathcal{A}^c} = 0.
\end{array}$$

From corollary 6.2 and from proposition 6.2, F^c is Leg_1-projectable. Moreover, it projects onto the integrable almost-product structure $(\mathcal{A}_1, \mathcal{B}_1)$ defined in example 4.2.

7. The second-order differential equation problem

As we know, a solution of the equation $i_X\omega_L = \mathrm{d}E_L$ (if it exists) is not necessarily a SODE, that is a vector field X on TQ such that $JX = C$. If L is almost regular, in [9, 11], Gotay and Nester have constructed a submanifold S of the final constraint submanifold P_f on which there exists a vector field ξ such that

$$\left(i_\xi\omega_L = \mathrm{d}E_L\right)_{/S} \qquad (J\xi = C)_{/S}\,. \tag{6}$$

By introducing a suitable almost-product structure on P_f we shall construct a submanifold S of P_f on which there exists an almost-product structure $(\mathcal{A}_S, \mathcal{B}_S)$ and a vector field ξ such that verifies (6) and, moreover, $\xi \in \operatorname{Im}\mathcal{A}_S$.

We first suppose that the presymplectic system (TQ, ω_L, E_L) admits a global dynamics. Consider an almost-product structure $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ adapted to $\ker\omega_L$ which is projectable onto an almost-product structure $(\mathcal{A}_1, \mathcal{B}_1)$ on M_1.

Remark 7.1. If ξ is any solution of equation of motion $i_X\omega_L = \mathrm{d}E_L$ then $\tilde{\mathcal{A}}(\xi)$ is a solution, too. Moreover, if the almost-product structure is *Leg*-projectable then $\tilde{\mathcal{A}}(\xi)$ is projectable onto $\mathcal{A}_1 Z$, where Z is any solution of the equation $i_X\omega_1 = \mathrm{d}h_1$.

From remark 7.1, given a vector field ξ on TQ which is a solution of the equation of motion

$$i_\xi\omega_L = \mathrm{d}E_L$$

then the vector field $\tilde{\mathcal{A}}(\xi)$ is projectable onto $\mathrm{A}_1(Z)$ and both are solutions of their respective equations of motion. As in [9, 11], there exists a unique point x in each fibre of $Leg_1 : TQ \longrightarrow M_1$ (where $Leg = i_1 \circ Leg$), such that $\tilde{\mathcal{A}}(\xi)$ verifies the second-order differential equation (SODE) condition at x, i.e. $(J(\tilde{\mathcal{A}}(\xi)))_x = C_x$. Consider the subset

$$S = \{x \in TQ \;/\; (J(\tilde{\mathcal{A}}(\xi)))_x = C_x\}\,. \tag{7}$$

In local coordinates, if $\tilde{\mathcal{A}}(\xi)$ is locally written as

$$\tilde{\mathcal{A}}(\xi) = \Xi^A\frac{\partial}{\partial q^A} + \tilde{\Xi}^A\frac{\partial}{\partial \dot{q}^A}\,.$$

Then, if $z = Leg_1(x) \in M_1$, and we identify z with the fibre which contains x, we deduce that Ξ^A is constant along the fibre. Moreover,

$$U = J(\tilde{\mathcal{A}}\xi) - C = (\Xi^A - \dot{q}^A)\frac{\partial}{\partial \dot{q}^A}$$

is tangent to the fibres. Let $\sigma(t) = (q^A(t), \dot{q}^A(t))$ be the integral curve of U which contains the point x with coordinates $(q_0^A, \dot{q}_0^A)$. We deduce that

$$\sigma(t) = (q_0^A, \Xi^A - e^{-t}(\Xi^A - \dot{q}_0^A))\,.$$

Then, we obtain

$$\bar{x} = \lim_{t\to\infty}\sigma(t) = (q_0^A,\ \Xi^A)\,.$$

Thus, the point $\bar{x}$ with coordinates (q_0^A, Ξ^A) is in the same fibre than x, since the fibres are closed. Moreover, $U(\bar{x}) = 0$, and, therefore, $\tilde{\mathcal{A}}(\xi)$ verifies the SODE condition at the point $\bar{x}$.

We obtain a differentiable section $\sigma : M_1 \longrightarrow TQ$ of Leg_1 and its image $S = \sigma(M_1)$ is a submanifold of TQ, on which $\tilde{A}(\xi)$ verifies the SODE condition. In general, $\tilde{A}(\xi)$ is not tangent to S, but the vector field $T\sigma(\mathcal{A}_1(Z))$ is tangent to S, it is a solution of the equation

$$(i_X\omega_L = \mathrm{d}E_L)_{/s}$$

and also satisfies the SODE condition. Moreover, since $\sigma : M_1 \longrightarrow S$ is a diffeomorphism, the almost-product structure $(\mathcal{A}_1, \mathcal{B}_1)$ on M_1 induces an almost-product structure $(\mathcal{A}_S, \mathcal{B}_S)$ on S such that for any solution of equation ξ_S

$$(i_X\omega_L = \mathrm{d}E_L)_{/s} \,. \tag{8}$$

We have that

$$\mathcal{A}_S(\xi_S) = T\sigma(\mathcal{A}_1(Z)) \,.$$

Summarizing, we have obtained the following result:

Proposition 7.1. Let ξ be any solution of the equation of motion

$$i_\xi\omega_L = \mathrm{d}E_L$$

and $(\tilde{A}, \tilde{B})$ an almost-product structure adapted to ω_L which is Leg-projectable onto an almost-product structure on M_1 and let S be the submanifold defined in (7). Then:

(i) There exists an almost-product structure $(\mathcal{A}_S, \mathcal{B}_S)$ adapted to the restriction of ω_L to S.
(ii) If ξ_S is any solution of (8) then $\mathcal{A}_S(\xi_S)$ is a solution which verifies the SODE condition.

In the general case, we apply the Gotay–Nester algorithm to the presymplectic system given by (TQ, ω_L, E_L). If the algorithm stabilizes, we denote by P_f, the final constraint submanifold. Consider on P_f an almost-product structure adapted to $\ker\omega_L \cap TP_f$ which is projectable onto M_f. Hence, by using a similar procedure to that used in proposition 7.1, we can obtain an almost-product structure $(\mathcal{A}_S, \mathcal{B}_S)$ on S adapted $\ker\omega_L \cap TS$ and a unique solution of the equation of motion $i_X\omega_L = \mathrm{d}E_L$ tangent to S which also verifies the SODE condition. Moreover, that solution belongs to $\mathrm{Im}\,\mathcal{A}_S$. We can also consider the equation:

$$i_X\omega_{P_f} = d(E_L)_{/P_f}$$

where $\omega_{M_f} = j_f^*\omega_L$, being $j_f : P_f \longrightarrow TQ$ the canonical embedding. Let $(\mathcal{A}_f, \mathcal{B}_f)$ be an almost-product structure adapted to $\ker\omega_{P_f}$ which is projectable to M_f (the final constraint submanifold on the Hamiltonian side). Then, from proposition 7.1 we obtain an almost-product structure adapted to ω_S where $\omega_S = j_S^*\omega_L$, being $j_S : S \longrightarrow P_f$ the canonical embedding. Moreover, if ξ is a solution of the equation

$$i_X\omega_S = j_s^*\mathrm{d}E_L$$

then $\mathcal{A}_S(\xi_S)$ is also a solution and verifies the SODE condition.

8. Affine Lagrangians on the velocities

In this section, we consider a particular case of degenerate Lagrangians: affine Lagrangians on the velocities. We study the almost-product structures adapted to ω_L that, in fact, are the Ehresmann connections in TQ. As in section 4.1, by using the second-class constraints, we construct an almost-product structure on T^*Q which gives the 'admissible' dynamic on the Hamiltonian side. The second-order differential equation problem is also studied.

An affine Lagrangian on the velocities L on TQ

$$L(q^A, \dot{q}^A) = \mu_A(q)\dot{q}^A + f(q^A)$$

may be globally defined as follows:

$$L = \hat{\mu} + f^V$$

where $\mu = \mu_A(q)\mathrm{d}q^A$ is a 1-form on Q and $f^V = f \circ \tau_Q$. Here $\hat{\mu} : TQ \longrightarrow \mathbb{R}$ denotes the function defined by:

$$\hat{\mu}(X_q) = \langle \mu(q), X_q \rangle \qquad \forall\, X_q \in T_qQ\,.$$

The energy and the Poincaré–Cartan forms are respectively

$$E_L = -f^V \qquad \alpha_L = -\mu^V \qquad \omega_L = \mathrm{d}\mu^V\,.$$

We have that $V(TQ) \subset \ker\omega_L$ and, hence

$$\dim\ker\omega_L \leqslant 2\dim(V(\ker\omega_L))\,.$$

L is a Lagrangian of type III according to the classification by Cantrijn *et al* [5].

Assume that the 2-form $\mathrm{d}\mu$ is symplectic. In that case, we have that $\ker\omega_L = V(TQ)$ and $(TQ, \mathrm{d}\mu^{\mathrm{v}}, -f^{\mathrm{v}})$ is a presymplectic system with a global dynamics. Consider an almost-product structure adapted to the presymplectic form ω_L. Then, we are giving a complementary of the vertical distribution, in other words, a connection in the tangent bundle TQ (see [18]). Given a connection Γ in TQ denote by h the horizontal projector and by v the vertical projector, respectively.

Since $\mathrm{d}\mu$ is symplectic, there exists a unique vector field X_f such that

$$i_{X_f}(-\mathrm{d}\mu) = \mathrm{d}f$$

that is, X_f is the Hamiltonian vector field with energy f. Since the complete lift X_f^c of X_f verifies that

$$i_{X_f^c}\omega_L = \mathrm{d}E_L\,. \tag{9}$$

Then, given a connection Γ, we fix a solution of (9) by taking $h(X_f^C) = X^C_{f\,E_L,h} = X_f^H$, which is the horizontal lift of X_f with respect to Γ.

The almost-product structure defined by the projectors (h, v) of the connection will define a Poisson bracket on TQ if and only if the horizontal distribution Im h is integrable, that is, if the connection is flat.

The Legendre transformation is given by

$$\begin{aligned} Leg:\ \ & TQ \longrightarrow T^*Q \\ & (q^A, \dot{q}^A) \longmapsto (q^A, \mu_A) \end{aligned}$$

and then the $M_1 = \operatorname{Im}\mu$. From proposition 6.1, the almost-product structure defined by (h, v) is projectable onto $Leg(TQ) = M_1$ because $\ker\omega_L = \operatorname{Im} v = V(TQ)$. We have that ω_1 is symplectic. Moreover, the map

$$\begin{aligned} \phi:\ \ & Q \longrightarrow Leg(TQ) = M_1 \\ & (q^A) \longmapsto (q^A, \mu_A) \end{aligned}$$

is a diffeomorphism and $Leg = \tau_Q \circ \phi$. Since $(\phi^{-1})^*\mathrm{d}\mu = \omega_1$ we get that ϕ is a symplectomorphism. From proposition 6.3, for each flat connection in TQ, the projection $\tau_Q : TQ \longrightarrow Q$ is a Poisson map, where we consider the Poisson bracket $\{\ ,\ \}_h$ on TQ and the Poisson bracket $\{\ ,\ \}_{\mathrm{d}\mu}$ defined by the symplectic form $\mathrm{d}\mu$ on Q.

All the primary constraints $\Phi_A = p_A - \mu_A$, $1 \leqslant A \leqslant m$, are second class since

$$\{\Phi_A, \Phi_B\} = \frac{\partial \mu_B}{\partial q^A} - \frac{\partial \mu_A}{\partial q^B}$$

and the matrix $(C_{AB}) = (\{\Phi_A, \Phi_B\})$ is regular because $\mathrm{d}\mu$ is symplectic. As in subsection 4.1, consider the almost-product structure $(\mathcal{P}, \mathcal{Q})$ on T^*Q given by the projector

$$\begin{aligned}\mathcal{Q} &= C^{AB} X_{\Phi_A} \otimes d\Phi_B \\ &= C^{AB}\left(\frac{\partial}{\partial q^A} - \frac{\partial \mu_A}{\partial q^C}\frac{\partial}{\partial p_C}\right) \otimes \left(\mathrm{d}p_B - \frac{\partial \mu_B}{\partial q^D}\mathrm{d}q^D\right).\end{aligned}$$

Then, $\operatorname{Im}\mathcal{P} = \ker \mathcal{Q}$ is generated by the vector fields

$$X_A = \frac{\partial}{\partial q^A} + \frac{\partial \mu_B}{\partial q^A}\frac{\partial}{\partial p_B} \qquad 1 \leqslant A \leqslant m$$

and, moreover, these vector fields are tangent to M_1.

From proposition 7.1, given a connection Γ in TQ we construct an m-dimensional submanifold S of TQ where there exists a solution of the equation

$$(i_X \omega_L = \mathrm{d}E_L)_{/S}$$

which verifies the SODE condition. Since (M_1, ω_1) is symplectic then (S, ω_S) is also symplectic and a straightforward computation gives us that $S = \operatorname{Im}(X^c)$ and the unique solution is precisely $\xi_S = X^c_{/S}$. Of course, it verifies the SODE condition. The vector field X_H also satisfies the SODE condition on S but, in general, it is not tangential to S.

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